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Application of full and simplified acoustic analogies to an elementary problem

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The axisymmetric impulse-heating case used by Tam in a critique of Lighthill's acoustic analogy is revisited. Linear and nonlinear sound levels are considered, as are three versions of the acoustic analogy: full; linearized; and one with the quadrupoles frozen at their initial distribution. Numerical solutions confirm that the full version reproduces the Euler solution and, in our analysis but contrary to Tam's, correctly identifies both the source of sound and nonlinear steepening. The linearized version is somewhat inaccurate at high sound levels, near 175 dB. The frozen version is almost as accurate as the linearized version at these levels, and identical at linear levels. In this admittedly artificial problem, it provides the radiated sound from the initial conditions only, i.e. without using the Euler solution. This is in contrast with the full acoustic analogy, which can be viewed as 'circular' in the sense of taking the history of all the Euler variables and returning the density.

1. Introduction and problem statement

Problems that have exact or effectively exact solutions are valuable to examine theories and approximations. Tam (2002) used three such problems to challenge the validity of the acoustic analogy (AA) (Lighthill 1952), both in terms of mathematics and in terms of the physical nature of the analogy and the search for the true sources of sound. The first problem was subsonic and axisymmetric in free space, and is studied here. The second involved a periodically heated sphere, and Tam found that the AA returned the correct answer, but argued that the quadrupoles are "fictitious noise source terms", whereas the "real source" is at the surface. This criticism is very similar to that levelled at the AA in the first test case, and addressed below. The charge derived from the third case was even more serious, since Tam challenged the status of the AA as an exact equation, using a simple normal shock wave. However, both Morris & Farassat (2003) and Peake (2004) demonstrated that Tam had overlooked a classic identity, which restores the consistency between the AA and the Rankine–Hugoniot relations. This leaves room only for differences of interpretation, between real and fictitious sources.

Consider the two-dimensional axisymmetric Euler equations, written for velocity, density and pressure:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho u)=0, \quad \frac{\partial}{\partial t}(\rho u)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho u^2)+\frac{\partial p}{\partial r}=0, \quad \frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}+\frac{\gamma p}{r}\frac{\partial}{\partial r}(ru)=0, \tag{1}$$

with the usual notation. Tam's initial conditions (Tam 2002) are

$$\rho = \rho_0, \qquad u = 0, \qquad p = p_0\left[1 + A\gamma \exp\left(-\log(2)\frac{r^2}{36L^2}\right)\right], \tag{2}$$

and the domain of interest is roughly $r \leqslant 100L$, $t \leqslant 100L/c_0$ where $c_0 = \sqrt{\gamma p_0/\rho_0}$. Tam set the perturbation level A to 0.4.

This initial condition amounts to impulsively heating fluid, at rest, near the origin. It then expands and sends a radial wave. The solution is linear in A for $A \ll 1$, both in the centre region and in the 'ring of sound'. This wave is not 'aerodynamic sound' in the usual sense of being a very weak by-product of turbulence with finite energy, but this case remains a valid exercise to interpret sound creation.

2. Numerical solutions and analysis via the full acoustic analogy

Solutions are obtained with second-order centred differencing and 500 points radially to $r/L = 100$; time integration is fourth-order accurate with 750 steps up to $t = 75$; t is normalized by L/c_0. The outer boundary condition at $r/L = 100$ is a zero perturbation, which is adequate as long as tc_0 is well below r. The results are numerically converged, as verified in grid-density and time-step studies, and very close to Tam's figure 2, as seen in figure 1(a) (for instance, at $t = 50$, Tam's figure shows the density peaking at 1.048 ± 0.0005 with $r = 57.2 \pm 0.2$, compared with 1.04743 and 57.37 here). Still, his curve at $t = 75$ appears slightly flawed near $r/L = 80$, reflecting some dispersion error due to a much coarser grid; his figure 5 also displays mild numerical ringing.

The acoustic analogy equation is here

$$\frac{\partial^2 \rho^*}{\partial t^2} - \frac{c_0^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \rho^*}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(r\rho u^2) + r\frac{\partial}{\partial r}\left(p - c_0^2\rho\right)\right], \tag{3}$$

where ρ on the right-hand side, or quadrupole source term, is known along with u and p (in the present studies, from a numerical Euler solution), while ρ^* on the left-hand side denotes the sound calculated via the AA. The time interval is $[0, \infty[$, and initial conditions are used; this differs from recent work of Morfey *et al.* (2006), which considered the interval $]-\infty, \infty[$ and generalized the AA to produce the jump at $t = 0$. The theory embodied in (3) could have two types of flaws: one leading to the outcome that $\rho^* \neq \rho$, and the other merely making the flow of information from the right-hand side to ρ^* physically misleading. Tam considered that he found instances of each among his examples.

Note that when in possession of complete analytical or numerically exact solutions, (3) could be written $\partial^2\rho^*/\partial t^2 - (c_0^2/r)\partial(r\partial\rho^*/\partial r)/\partial r = \partial^2\rho/\partial t^2 - (c_0^2/r)\partial(r\partial\rho/\partial r)/\partial r$, or any equivalent quantity. The standard form in (3) may have been picked by Lighthill on the merit of being in divergence form and free of time derivatives, which may make it physically the most relevant (such derivatives often re-appear when Green's functions are integrated by parts). It may also have been expected to make better use of partial knowledge of the flow field, either in spatial terms or in terms of available quantities, and to be more revealing of scaling with Mach number. This is speculation. A clear criterion to rule that the equation written in this paragraph is senseless does not appear to be available. This aside has nothing to do with Tam's point.

The initial conditions for (3) are $\rho^* = \rho_0$ and $\partial\rho^*/\partial t = 0$. The quadrupole term Q_{rr}, the right-hand side of (3), is shown in figure 2(a) which repeats Tam's figure 4, except

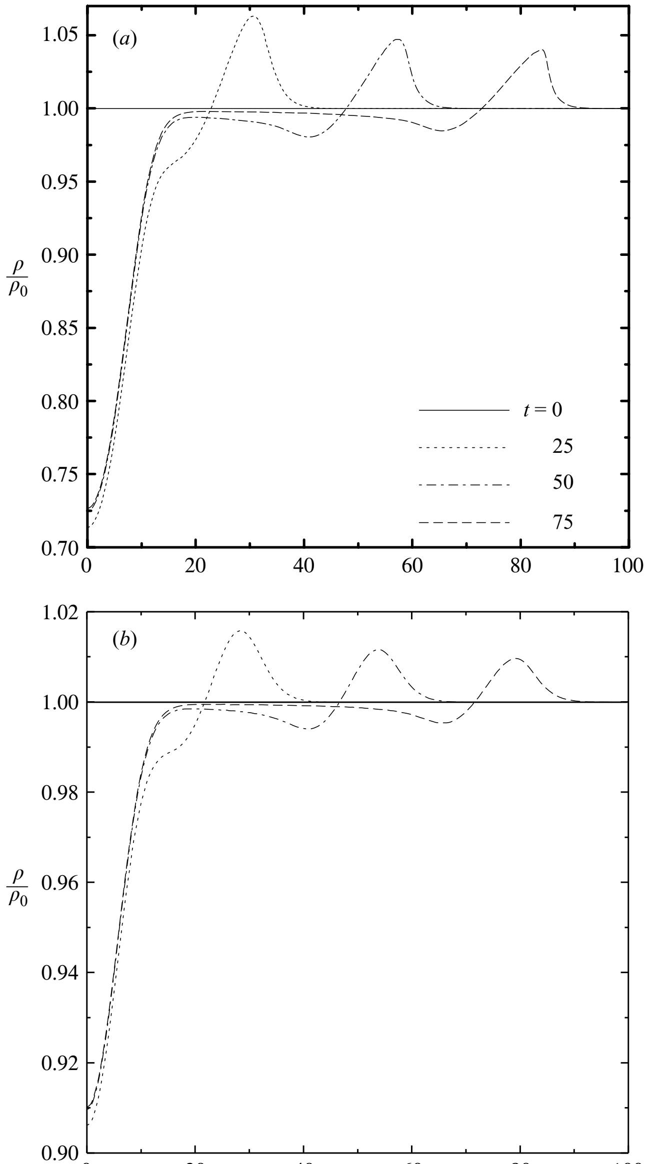

FIGURE 1. Density at different times, from the Euler solution. (*a*) $A = 0.4$; (*b*) $A = 0.1$.

that the initial distribution is also shown. The steady-state distribution for $t \to \infty$ has a dip in the centre, required to maintain the dip in ρ; both are $O(A)$ as $A \to 0$. The difference between the distributions at $t = 0^+$ and for $t \to \infty$ is a nonlinear effect. The 'ringing' part of the quadrupole which rides the wave peaks at -2.5×10^{-3} for $A = 0.4$, and -1.0×10^{-4} for $A = 0.2$, a ratio which is much larger than in the $O(A^2)$ scaling that prevails when $A \ll 1$ (both terms on the right-hand side of (3) being quadratic to leading order). The high value $A = 0.4$ comes close to creating a shock within 100 time units.

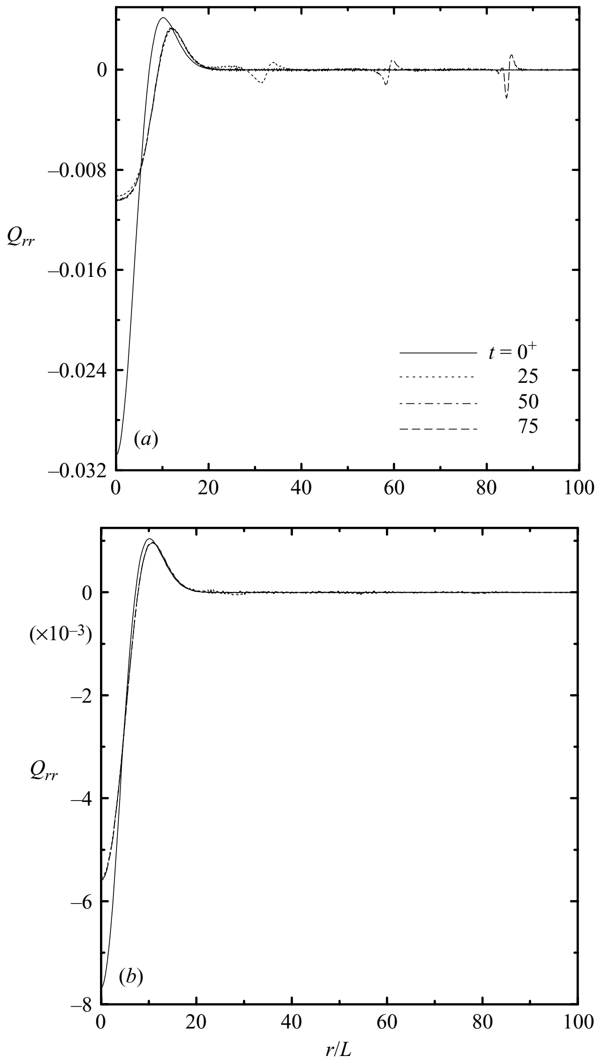

FIGURE 2. Quadrupole source term at different times, from the Euler solution. (*a*) $A = 0.4$; (*b*) $A = 0.1$.

The essential argument can now be made. Our reading of this figure is that the initial distribution reflects the pressure imbalance imposed on the fluid by (2), which vanishes and eventually allows a steady solution of (3), after the radial wave has escaped. The quadrupole distribution contains a tremor that follows the density wave, but this perturbation itself does not 'generate' the wave via the AA, as is obliquely implied by Tam (with figures drawing attention to a ring of quadrupoles in parallel to the ring of sound, followed by the words "The Lighthill quadrupole noise sources are only fictitious sources. The true noise source is the initial pressure pulse") and constitutes his objection. It only reflects subsequent steepening of the wave, and therefore its behaviour is not physically misleading. This is confirmed by the corresponding curves for $A = 0.1$ in figure 2(*b*), scaled with A for easy comparison with figure 2(*a*); the initial quadrupole term is linear in A, but its propagating ring is

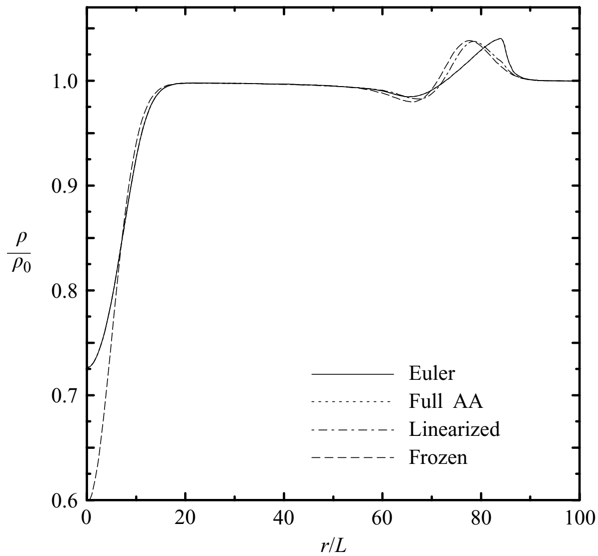

FIGURE 3. Density at $t = 75$ from the Euler solution and three versions of the AA.

not. The nonlinear effect has nearly vanished, and the physical cause of the sound is indeed near the origin, both in space and time.

3. Simplified versions of the acoustic analogy

Much like the amplitude variations, these help distinguish issues more clearly. In the linearized version of the AA, the velocity term proportional to u^2 is omitted in (3), leaving only $(1/r)\partial[r\partial(p - c_0^2\rho)/\partial r]/\partial r$. This quantity is again obtained from a solution of the Euler equations. The 'frozen' version is motivated by the observation that, to leading order, $\partial p/\partial t = c_0^2\partial\rho/\partial t$. This suggests solving (3) with $(p - c_0^2\rho)$ left in its state at $t = 0^+$, obtained from (2), obviating the need for an Euler solution and therefore the 'circular' nature of the AA (Crow 1970). This version of the AA is predictive on its own. Its result could be obtained by a numerical quadrature, as opposed to solving the partial differential equation (3). A mental extrapolation to $A \ll 1$ from figures 2(*a*) and 2(*b*) confirms that at linear levels, the quadrupole indeed becomes time-independent.

The results in figure 3, again for the nonlinear level $A = 0.4$, show that the evolution of the sound wave into a near-N wave is suppressed by the linearization, as expected, but that the peak sound pressure is not grossly under-predicted. The dotted line of the full AA is covered by the solid line of the Euler result. More surprising perhaps is the outcome of the frozen-quadrupole solution, which is close to the true linearized version in spite of the sizable evolution seen in figure 2(*a*). Where this version fails is near the origin. Nevertheless, it is an intriguing thought that the AA produced the linear acoustic effect, using only knowledge of the initial condition. This confirms that the AA correctly identifies the origin of the sound as being the heating and associated pressure imbalance at $t = 0$.

4. Summary

Re-visiting a simple problem of sound creation and propagation led to a view of the acoustic analogy that is much more favourable than Tam's, thanks to a more comprehensive display of the quadrupole distribution and to separating nonlinear effects. The same problem in three dimensions leads to the same finding (results not shown here); the nature of Tam's heated-sphere case also strongly suggests that the same conclusion would be reached. Conversely, it is indisputable that using a velocity a_0 different from c_0 in (3) would create an apparent source $(a_0^2 - c_0^2)/r \ \partial[r\partial\rho/\partial r]/\partial r$, now linear in A, which tracks the wave just like the nonlinear term does in figure 2(*a*). This would revive Tam's objection, but only if an 'un-natural' speed of sound were to be considered acceptable.

Additionally, two simplified versions of the AA give valuable results, within their logical limits, in this case which remains quite artificial compared with genuine situations of noise generated by turbulence (usually, the 'Reynolds-stress' term $\partial(r\rho u^2)/\partial r$ dominates the 'entropy' term $r\partial(p - c_0^2\rho)/\partial r$).

It appears that demonstrating that the acoustic analogy gives misleading indications in a simple case has not been achieved; this parallels Morris & Farassat's and Peake's findings. This is not to state that the AA is immune to any criticism, but it requires more depth and more complex flows, as for instance in Crow's (1970) study.

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